Online Algorithms: going beyond the worst-case

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NYU COURANT

analysis of algorithms

reigning paradigm: **worst-case analysis** of algorithms

how does the algorithm perform on its worst-possible input?

analysis of algorithms

reigning paradigm: **worst-case analysis** of algorithms

+ robustness

+ wide applicability

+ many algorithms with good worst-case bounds

+ often less contentious

Naturally, there are shortcomings as well…

- pessimism, and insensitivity to data model/predictions

analysis of algorithms

Ideally: want to get algorithms that are good for

worst-case ***and*** "best-case" ***and*** …. all cases.

Worst-case: robustness when data is unpredictable

"Best-case": efficiency when data follows anticipated patterns

How to go beyond the worst case?

let's see glimpse of ideas/techniques in context of online algos

Online Algorithms

Requests arrive over time, must be served immediately/irrevocably

Goal: (say) minimize cost of the decisions taken

Competitive ratio of algorithm A:

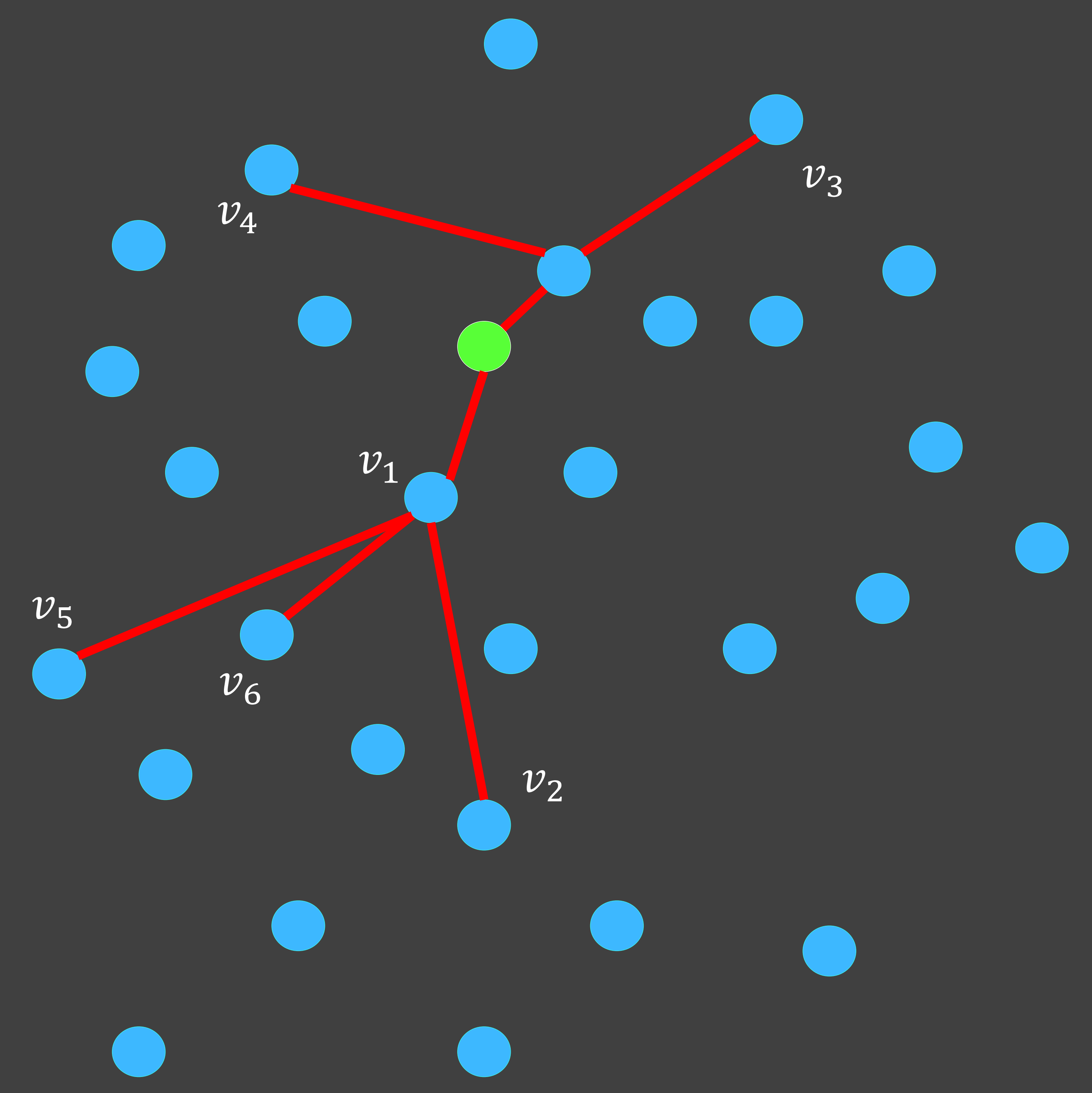
$$\max_{\text{instances } I} \frac{\text{cost of algorithm } A \text{ on instance } I}{\text{optimal cost to serve } I}$$

Want to minimize the competitive ratio.

Online (Steiner) Tree

Metric space. n points arrive over time, maintain a connected tree.

Goal: minimize cost of tree



Competitive ratio of algorithm A:

$$\max_{\text{instances } I} \frac{\text{cost of algorithm } A \text{ on instance } I}{\text{optimal cost to serve } I}$$

Want to minimize the competitive ratio.

Online Set Cover

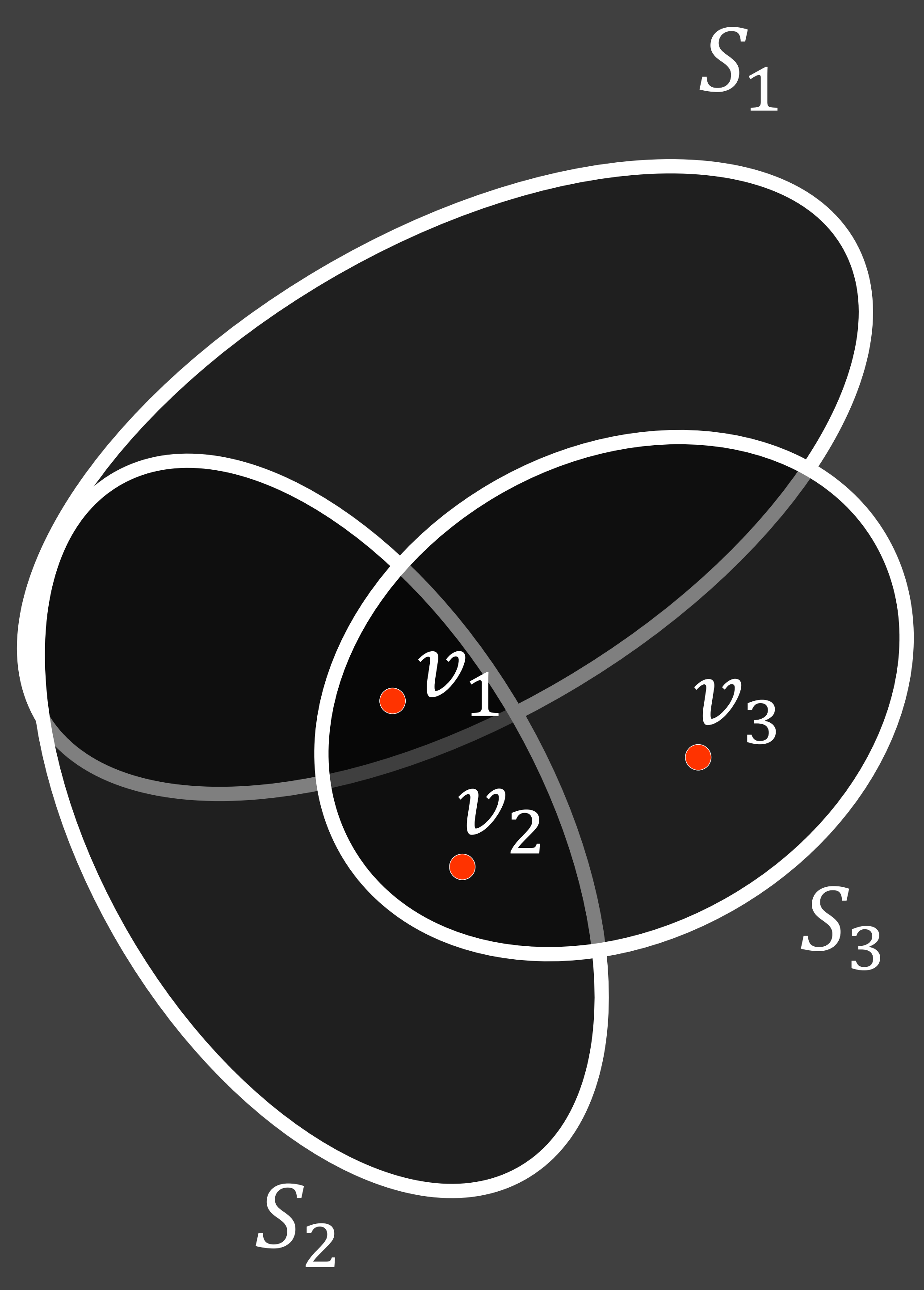
Set system. n elements arrive over time, want to maintain a cover.

Goal: minimize cost of sets picked

Competitive ratio of algorithm A:

$$\max_{\text{instances } I} \frac{\text{cost of algorithm } A \text{ on instance } I}{\text{optimal cost to serve } I}$$

Want to minimize the competitive ratio.



max-K finding

n people arrive over time, each has value v_i -- can pick at most K

Goal: (say) **maximize** sum of values of picked people

Competitive ratio of algorithm A:

$$\min_{\text{instances } I} \frac{\text{value of algorithm } A \text{ on instance } I}{\text{optimal value on instance } I}$$

$$\max_{\text{instances } I} \frac{\text{cost of algorithm } A \text{ on instance } I}{\text{optimal cost on instance } I}$$

Want to **maximize** the competitive ratio.

Want to minimize the competitive ratio.

price of uncertainty

	Offline	Online
Steiner tree	~ 1.3	$\Omega(\log n)$
Set Cover	$O(\log n)$	$\Omega(\log^2 n)$
Max-K-find	1	$O(K/n)$

can we do better in non-worst-case settings?

today's menu

models to go beyond worst-case:	max-find, spanning tree, set cover
but don't overfit to these models:	max-k-finding
and perhaps use predictions…:	paging/caching

price of uncertainty

	Offline	Online
Steiner tree	~ 1.3	$\Omega(\log n)$
Set Cover	$O(\log n)$	$\Omega(\log^2 n)$
Max-K-find	1	$O(K/n)$

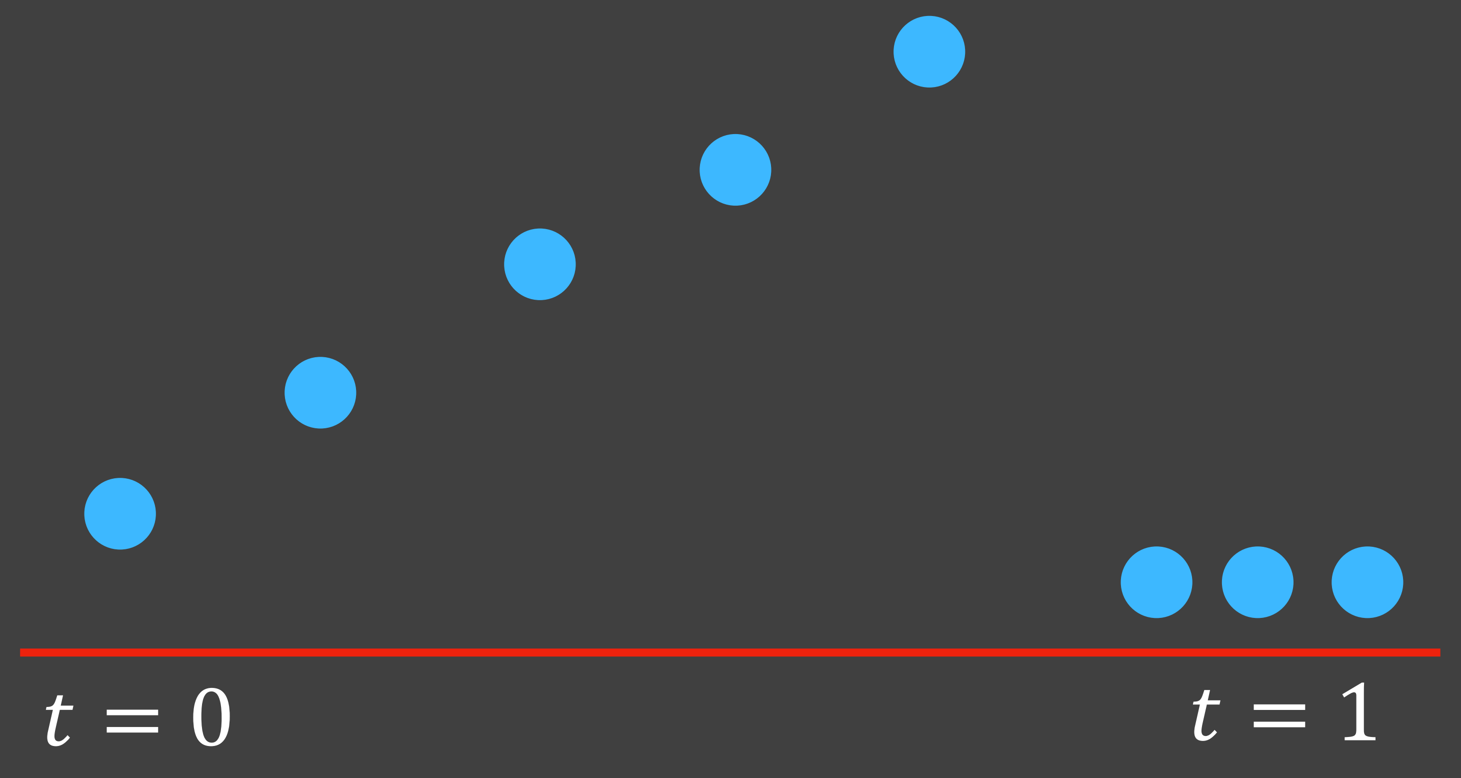
max-1 finding

n people arrive over time, each has value v_i -- pick at most one

Goal: maximize value of picked person

worst-case instance:

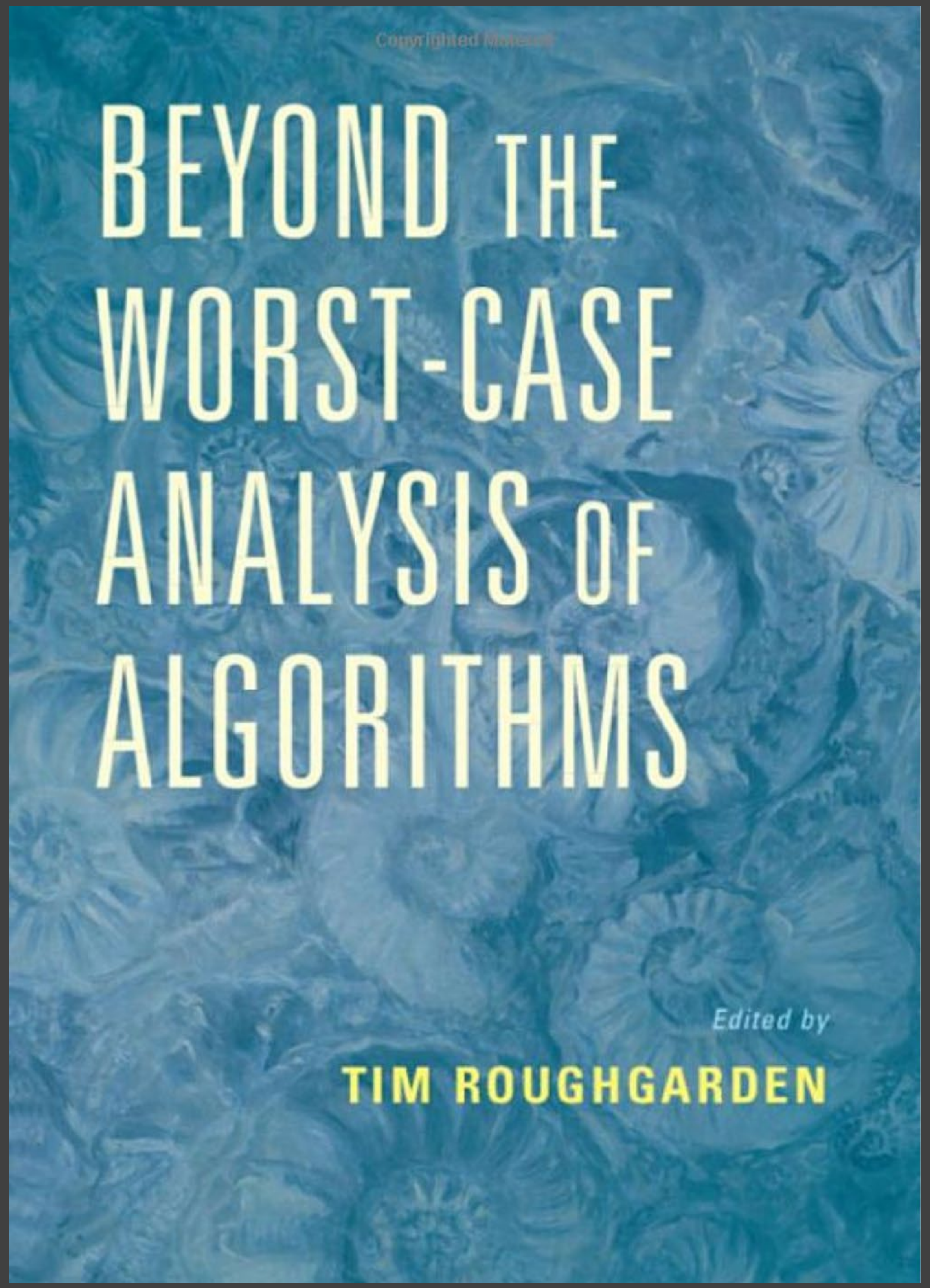
random guessing is the best option here
$\Rightarrow 1/n$ chance of success



going beyond the worst case

Ways to model non-worst-case instances?

1. values in bounded range
2. draws from some stochastic process? (Say $v_i \sim \mathcal{D}_i$)
3. maybe arrival order is not worst-case?
4. train NN to find patterns, give predictions
5. …



prophet model

[Krengel and Sucheston 78]
[Kesselheim 17] [Rubinstein Wang Weinberg 20]

n items arrive online, have value $R_i \sim \mathcal{D}_i$ (indep.) Distributions $\mathcal{D}_i$ known.

Pick one item. Maximize (expected) value.

Algorithm:

Take one sample $S_1, S_2, \ldots, S_n$ from each distribution.

Set threshold $T \leftarrow$ their maximum

Pick first R_i above threshold T

Thm: $\mathbb{E}[Alg] \geq {}^1/_4 \; \mathbb{E}[\max_i R_i]$

prophet model

[Krengel and Sucheston 78]
[Kesselheim 17] [Rubinstein Wang Weinberg 20]

Can get $^1/_2$!!

Thm: $\mathbb{E}[Alg] \geq {}^1/_4 \ \mathbb{E}[\max_i R_i]$

Samples $S_1, \dots, S_n$

Real values $R_1, \dots, R_n$

sort

$W_1 > W_2 > \cdots > W_{2n}$

Pr[W_1 *is real*] = ½

Pr[W_2 *is sample* | W_1 *real*] ≥ ½

⇒ Pr[$W_1 = R_{\max}$ *chosen*] ≥ ¼

[Flood 49, Gardner 60, Dynkin 63]

secretary model

n items have values chosen by adversary. But arrive online in random order.

Pick one item. Maximize (expected) value.

Algorithm:

Ignore first $^1/_2$ fraction of items.

Set threshold $T \leftarrow$ their maximum

Pick first item among remaining above threshold T

Can get $^1/_e$!!

Thm: $\mathbb{E}[Alg] \geq {}^1/_4 \; OPT$

[Dütting Lattanzi Paes Leme Vassilvitskii 21]

algos with predictions

Train a classifier to predict if current item is maximum among remaining

Model: like sec'y, but each prediction correct w.p. $p \geq {}^1/_2$ independently

Algo: ignore some fraction of elements

- then (for some fraction) pick any item that is best so far, and predictor = "Yes"
- then (for remaining fraction) pick any item that is best so far (ignore predictor)

Theorem: optimal performance for this model.

price of uncertainty

	Offline	Online	BWC
Steiner tree	~ 1.3	$\Omega(\log n)$	
Set Cover	$O(\log n)$	$\Omega(\log^2 n)$	
Max-K-find	1	$O(K/n)$	$\Omega(1)$ prophet, RO

rest of today's menu

models to go beyond worst-case:	spanning tree and set cover
but don't overfit to these models:	max-k-finding
and perhaps use predictions…:	paging/caching

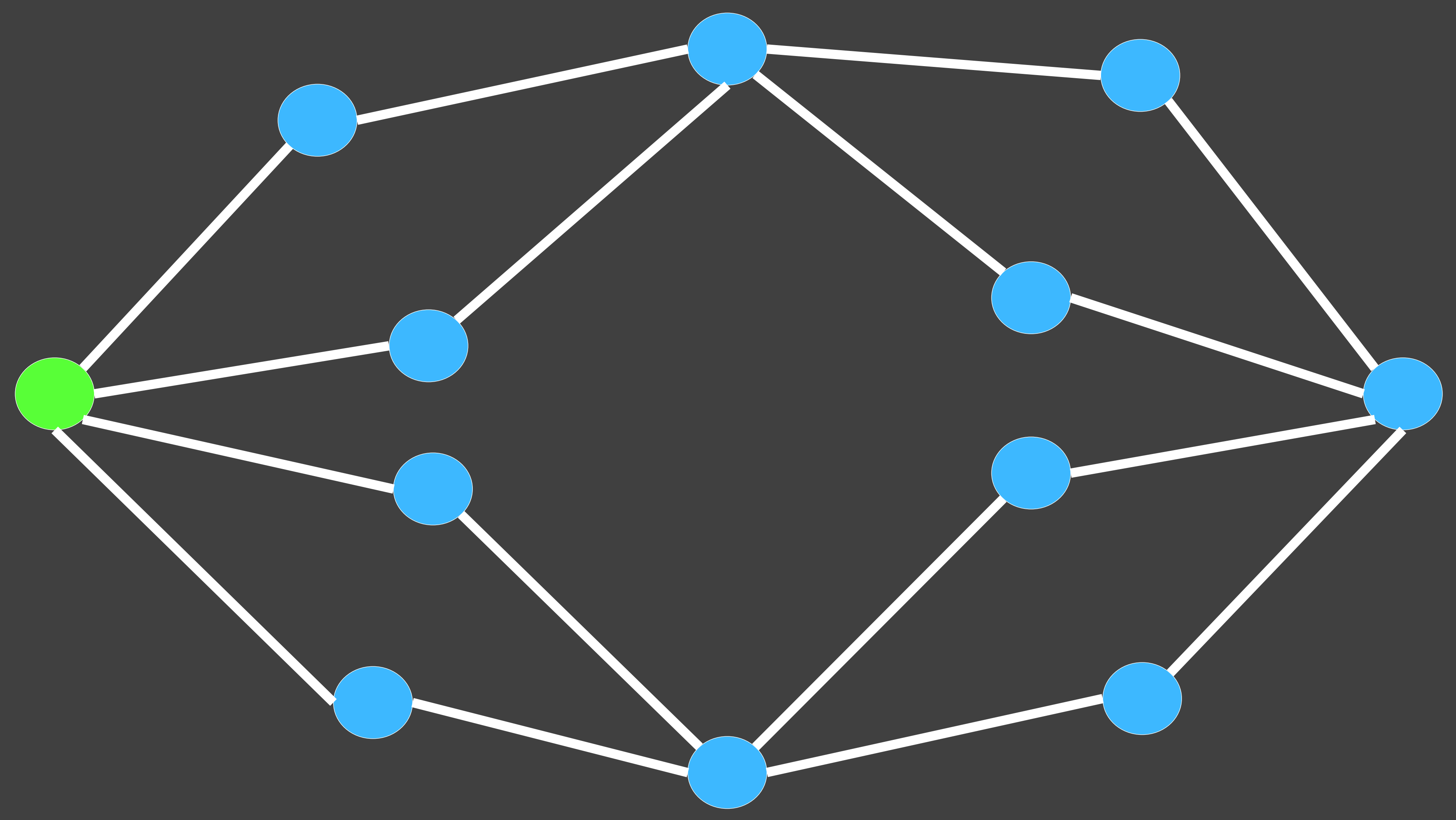
[Imase Waxman 91]

online (steiner) tree

Suppose n requests

Connect each request on arrival

Worst-case comp.ratio: $\Theta(\log n)$



Goal: minimize total cost of edges

[Garg G. Leonardi Sankowski 08]

prophet (steiner) tree

Suppose n requests: vertex $R_i \sim \mathcal{D}_i$

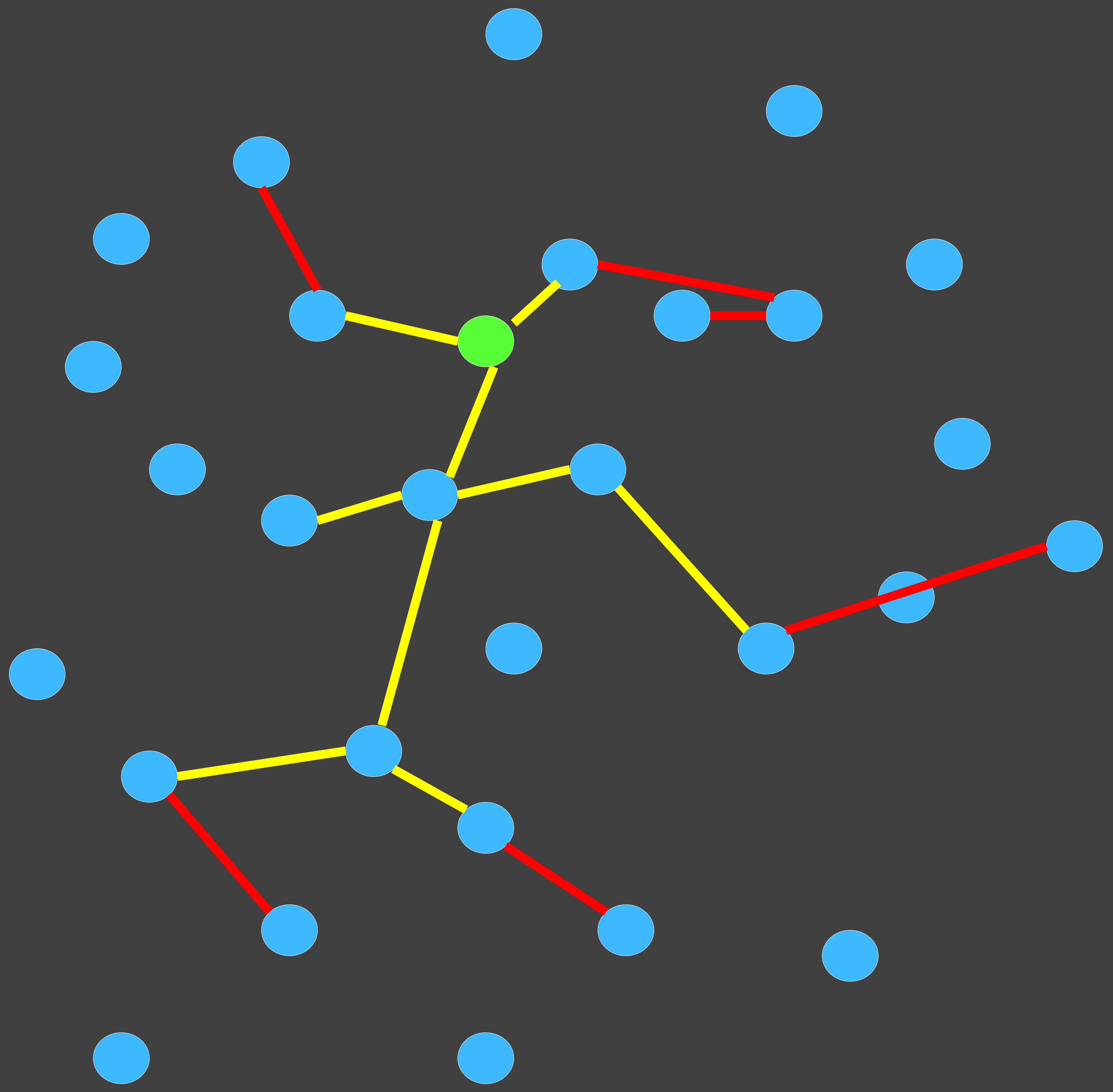
Connect each request on arrival

Algorithm:

For all i, take one sample $S_i \sim \mathcal{D}_i$ each

Build MST on $S_1, \ldots, S_n$

When actual requests $R_i \sim \mathcal{D}_i$ arrive:
connect to closest previous point



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Build MST on $S_1, \dots, S_n$

When actual requests $R_i \sim \mathcal{D}_i$ arrive:
connect to closest previous point

Theorem: $\mathbb{E}[Algo] \le 2\ \mathbb{E}[MST(R_1, \dots, R_n)]$

Proof: $\mathbb{E}[MST(S_1, \dots, S_n)] = \mathbb{E}[MST(R_1, \dots, R_n)]$

$$\mathbb{E}[cost(R_i)] \le \mathbb{E}[dist(R_i, S)]$$
$$\le \mathbb{E}[dist(R_i, S_{-i})]$$
$$= \mathbb{E}[dist(S_i, S_{-i})]$$

$$\Rightarrow\ \Sigma_i\ \mathbb{E}[cost(R_i)] \le \Sigma_i\ \mathbb{E}[dist(S_i, S_{-i})] \le \mathbb{E}[\ MST(S)\ ]$$

price of uncertainty

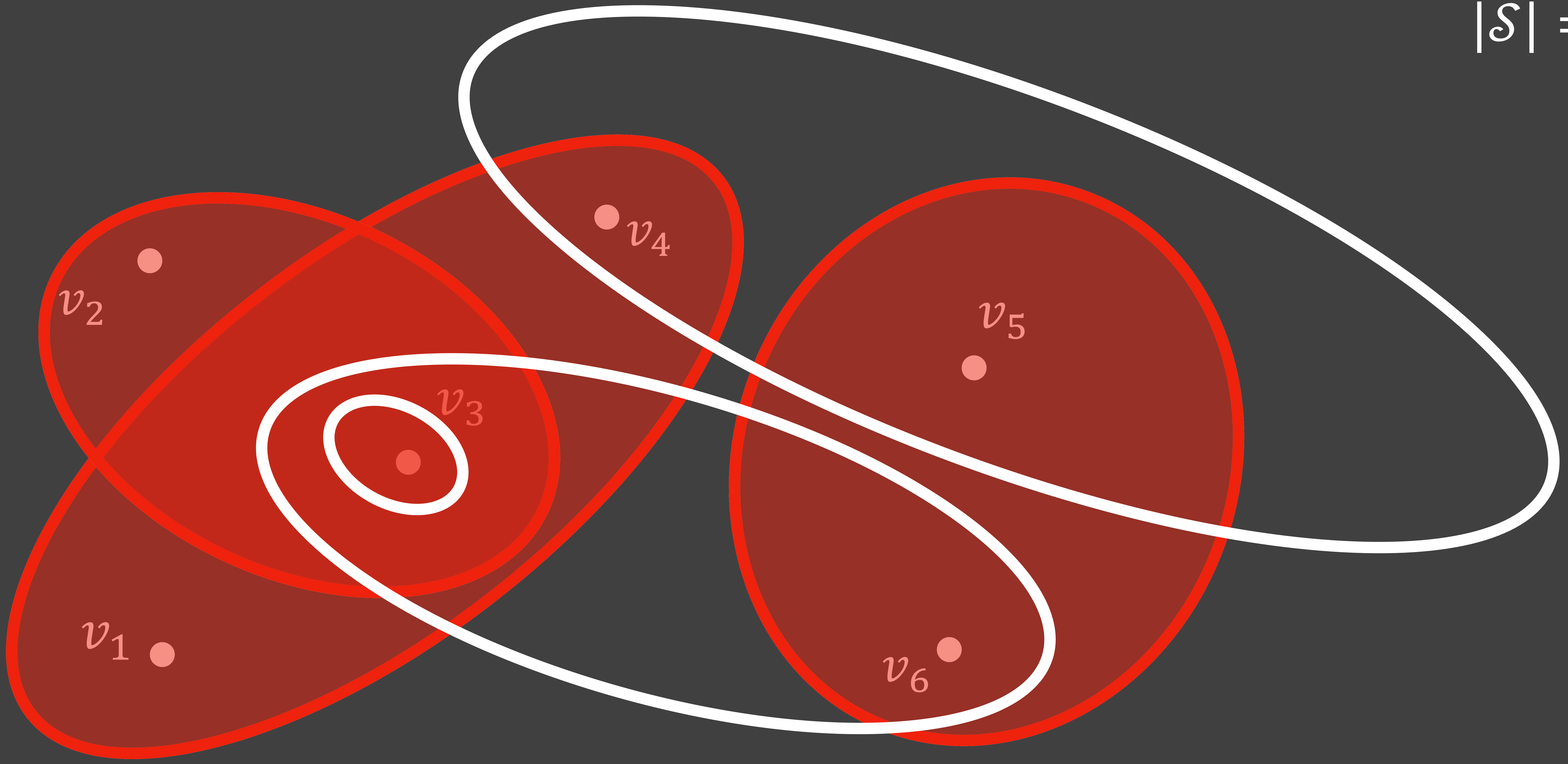
	Offline	Online	BWC
Steiner tree	~ 1.3	$\Omega(\log n)$	2 prophet
Set Cover	$O(\log n)$	$\Omega(\log^2 n)$	
Max-K-find	1	$O(K/n)$	$\Omega(1)$ prophet, RO

[Alon Awerbuch Azar Buchbinder Naor 03]

Online Set Cover

$|\mathcal{U}| = n =$ # elements

$|\mathcal{S}| = m =$ # sets



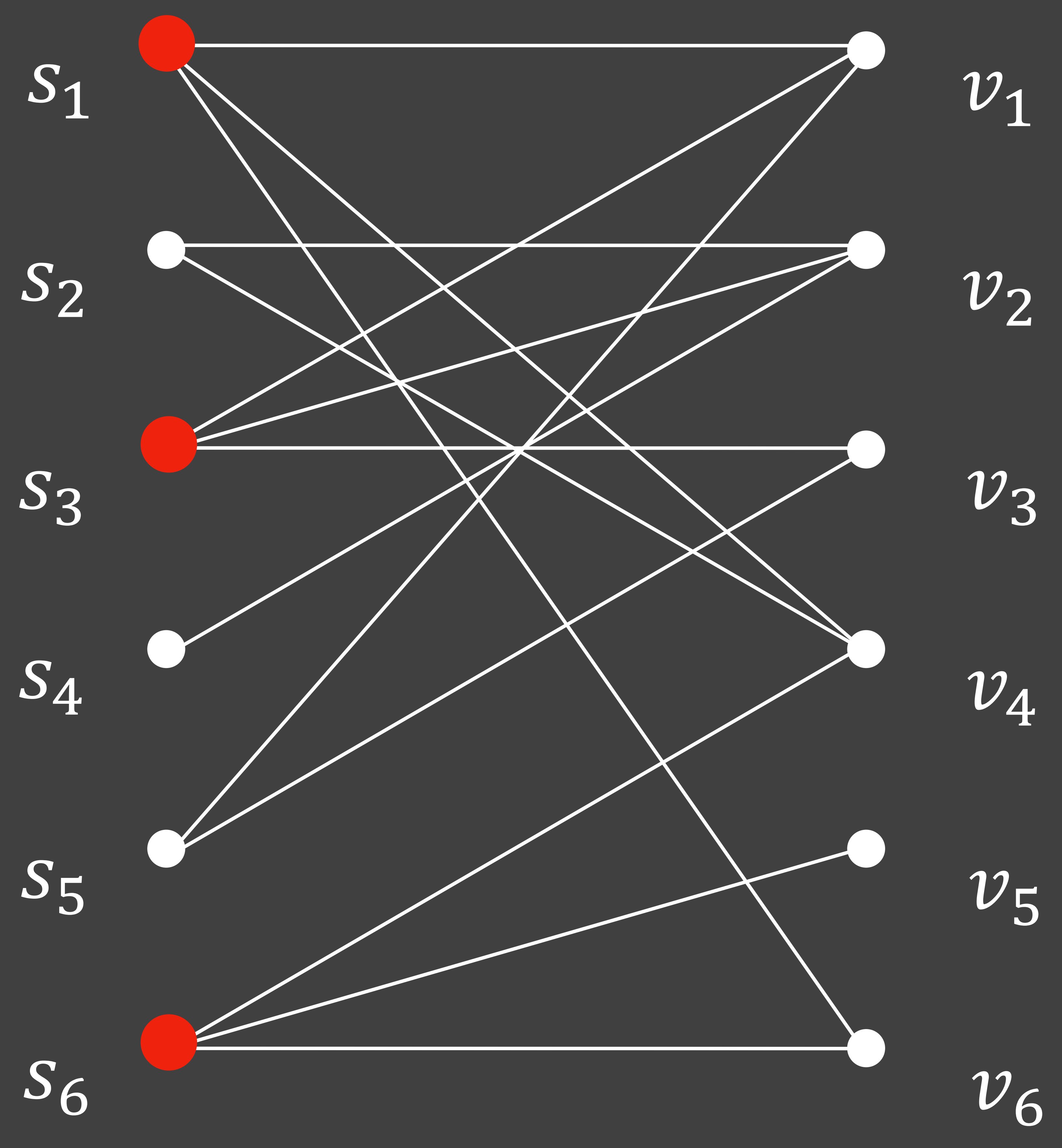
Goal: pick smallest # sets to cover all elements.

[Alon Awerbuch Azar Buchbinder Naor 03]

Online Set Cover

$\mathcal{F}$

m sets



$\mathcal{U}$

n elements

[Alon Awerbuch Azar Buchbinder Naor 03, Feige Korman 05]

Online Set Cover

Algorithm:
$O(\log n \log m)$
competitive

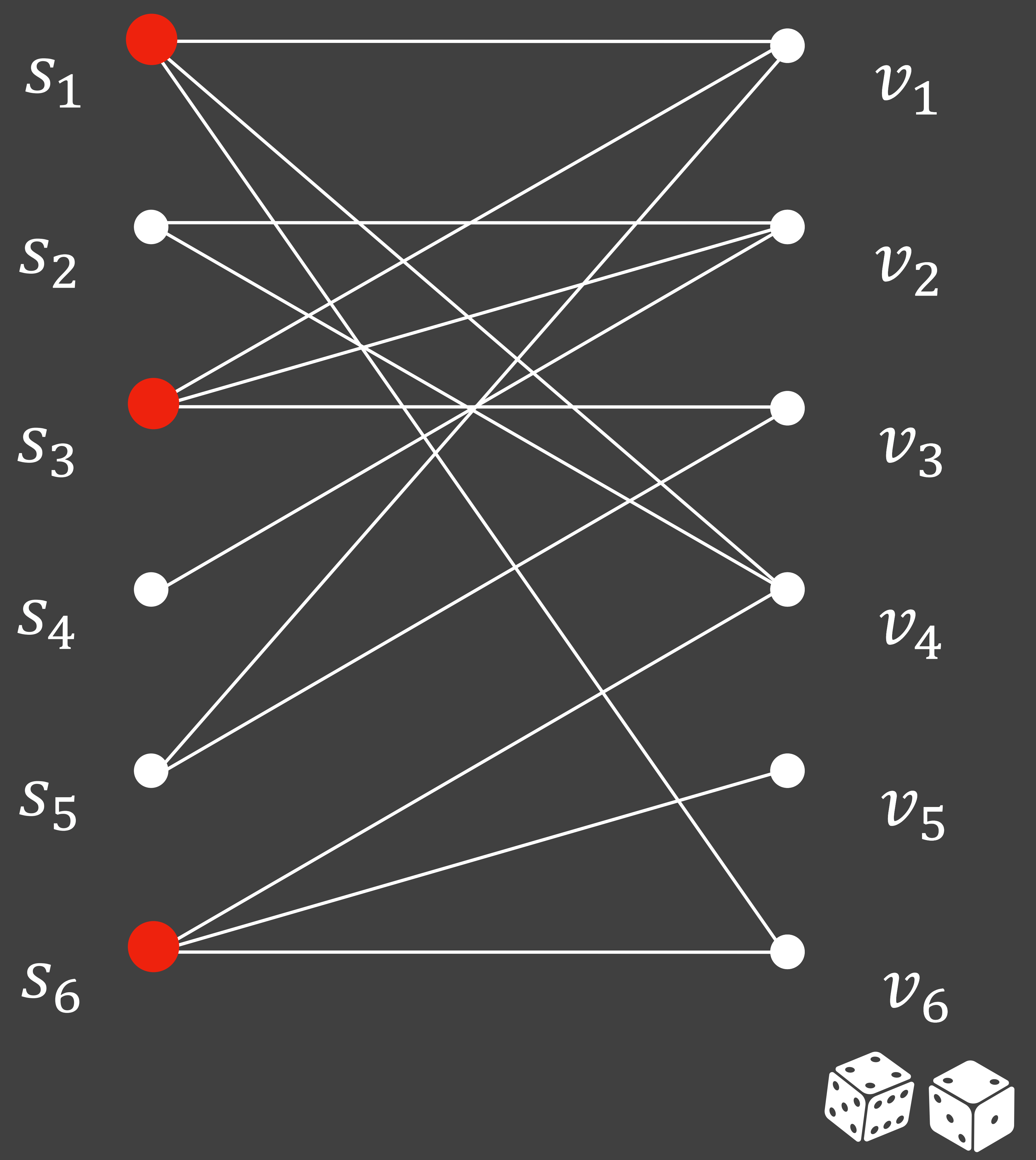
CR: $\Omega(\log n \log m)$
for deterministic algos
and for poly-time algos

Q: What happens beyond the worst case?

Random Order (RO)

$\mathcal{F}$

m sets



$\mathcal{U}$

n elements

[Gupta Kehne Levin FOCS 21]

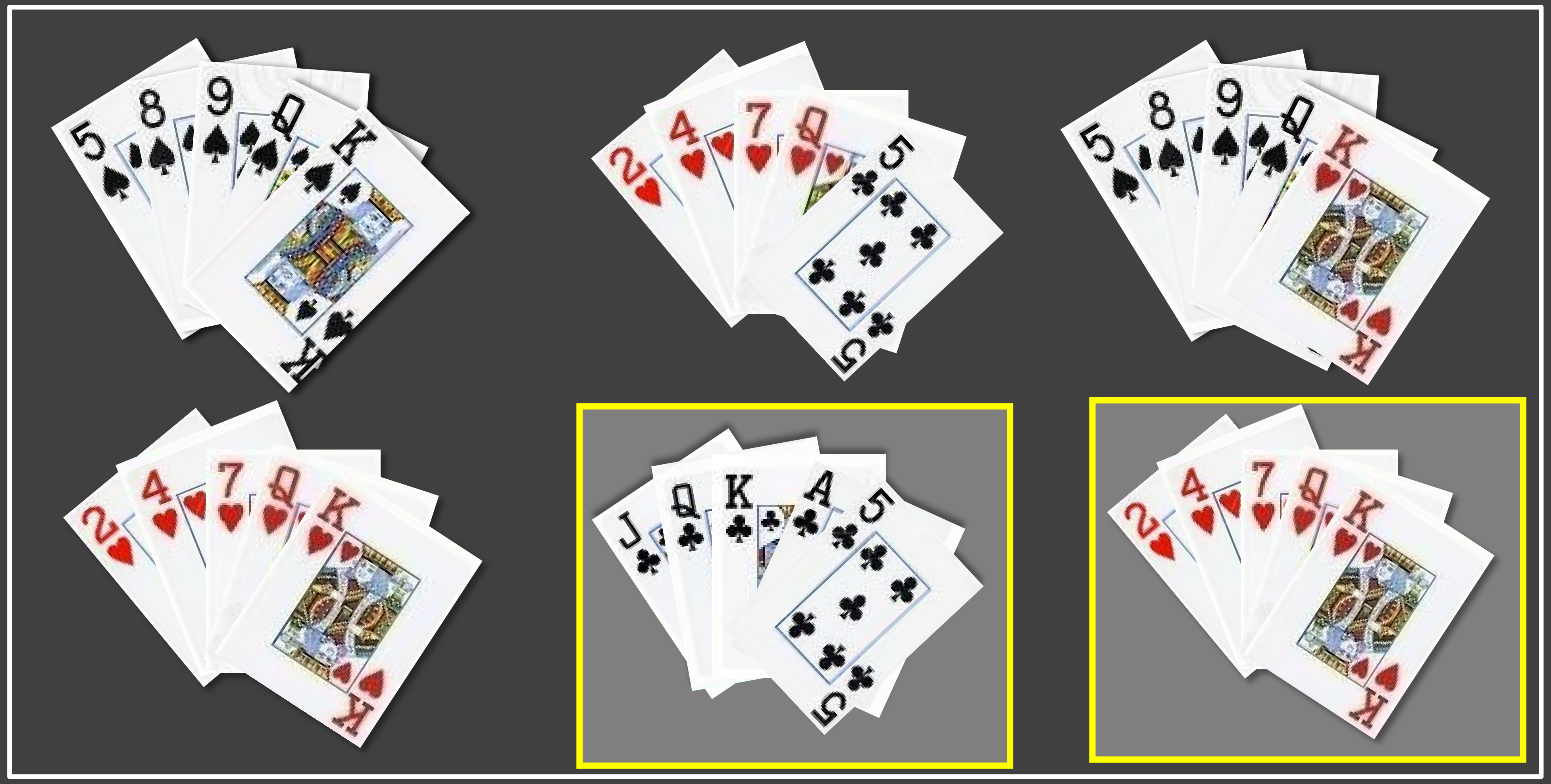
LearnOrCover

(Unit cost, exp time)

Assume we know k = OPT

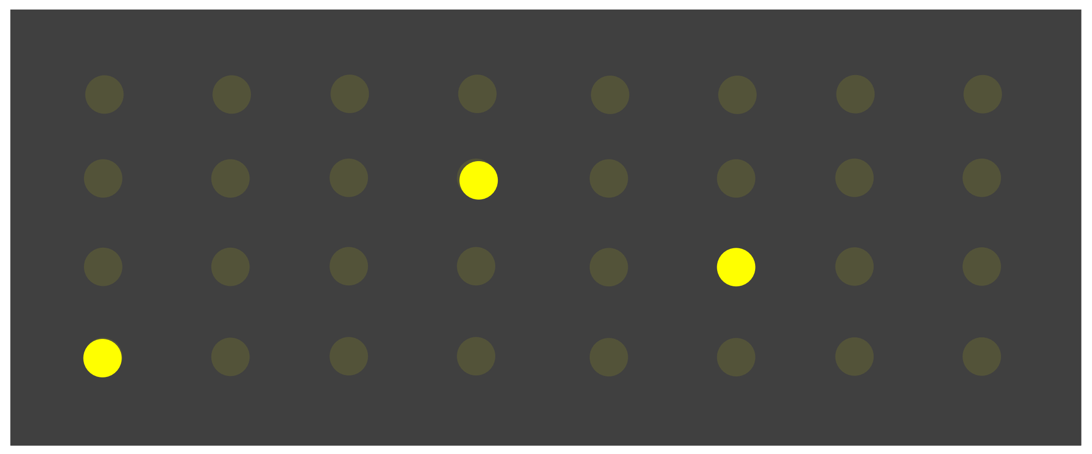
when random element v arrives
- if v not already covered, in parallel:
  1. select random remaining candidate
     pick random set from it
  2. remove candidates that don't cover v
     pick any set covering v

candidate solutions



$\mathcal{P}$

$\binom{m}{k}$



$\mathcal{U}$

Q: do ½ of remaining candidates cover ½ of uncovered elements?

Yes: random set covers many uncovered elements!

No: random element removes many candidates!!

Sol R:

[Gupta Kehne Levin FOCS 21]

Case 1: $\geq 1/2$ of $P \in \mathcal{P}$ cover $\geq 1/2$ of $\mathcal{U}$.

R covers $\frac{|\mathcal{U}|}{4k}$ in expectation.

$\mathcal{U}$ shrinks by $\left(1 - \frac{1}{4k}\right)$ in expectation.

$|\mathcal{U}|$ initially n

$\Rightarrow \quad O(k \log n)$ COVER steps suffice.

Case 2: $> 1/2$ of $P \in \mathcal{P}$ cover $< 1/2$ of $\mathcal{U}$.

$\geq 1/2$ of $P \in \mathcal{P}$ pruned w.p. $1/2$.

$\mathcal{P}$ shrinks by $3/4$ in expectation.

$|\mathcal{P}|$ initially $\binom{m}{k} \approx m^k$

$\Rightarrow \quad O(k \log m)$ LEARN steps suffice.

$\Rightarrow O(k \log mn)$ steps suffice.

[Gupta Kehne Levin FOCS 21]

LearnOrCover

(Unit cost)

Init. $x \leftarrow 1/m$.

@ time t, element v arrives:

If v covered, do nothing

Else:

(I) Buy random $R \sim x$.

(II) $\forall S \ni v$, set $x_S \leftarrow e \cdot x_S$

Renormalize $x \leftarrow x / \| x \|_1$

Buy arbitrary set to cover v

Idea: Measure convergence with potential function

$$\sum_S x_S^* \log \frac{x_S^*}{x_S^t}$$

$$\Phi(t) = c_1 \, KL(x^* || x^t) \; + c_2 \log |\mathcal{U}^t|$$

$\mathcal{U}^t$:= uncovered elements @ time t

x^* := uniform distribution on OPT

Claim 1: $\Phi(0) = O(\log mn)$, and $\Phi(t) \geq 0$.

Claim 2: If v uncovered, then $E[\Delta\Phi] \leq -\frac{1}{k}$.

If $\mathbb{E}_v[\, x_v \,] > \frac{1}{4} \Rightarrow \mathbb{E}_R[k \, \Delta \log |\mathcal{U}^t|]$ drops by $\Omega(1)$.

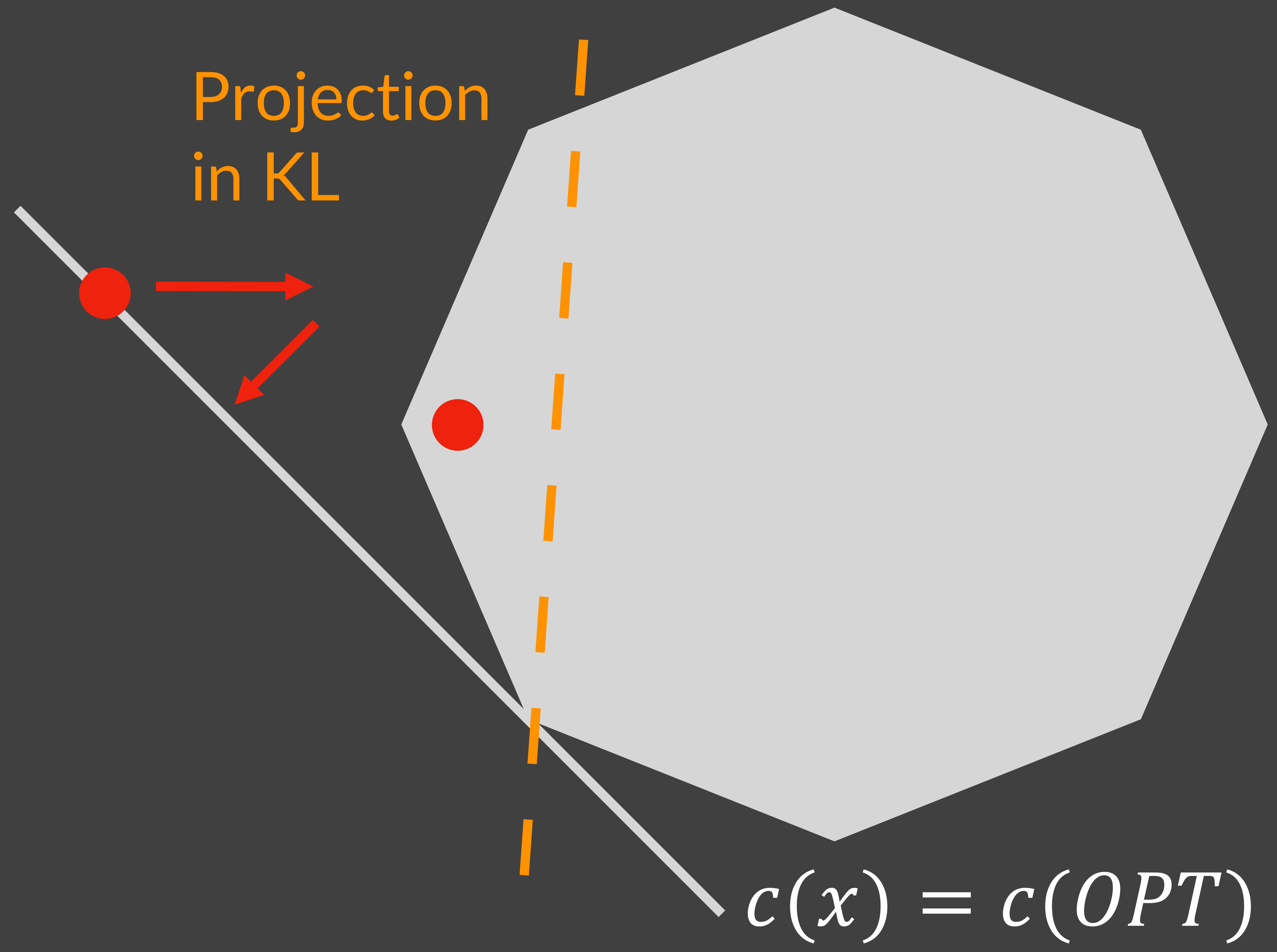
Else $\mathbb{E}_v[k \, \Delta KL]$ drops by $\Omega(1)$.

(Recall $k = |OPT|$)

LearnOrCover

(Some philosophy)

Perspective 1:



[Alon+ 03]
LearnOrCover
[Buchbinder G. Molinaro Naor 19]

Perspective 2:

Define

$$f(x) := \sum_v \max\left(0, 1 - \sum_{S \ni v} x_S\right)$$

(Goal is to minimize f in smallest # of steps)

$\nabla f|_S(x)$ = # uncovered elements in S

$$\propto E[\mathbb{1}\{v \in S \mid v \text{ uncovered}\}]$$

RO reveals stochastic gradient…

price of uncertainty

	Offline	Online	BWC
Steiner tree	~ 1.3	$\Omega(\log n)$	$O(1)$ prophet
Set Cover	$O(\log n)$	$\Omega(\log^2 n)$	$O(\log n)$ RO
Max-K-find	1	$O(K/n)$	$\Omega(1)$ prophet, RO

today's menu

models to go beyond worst-case:	max-finding, spanning tree, set cover
but don't overfit to these models…:	max-k-finding
and perhaps use predictions…:	paging/caching

robustness vs efficiency

worst-case

very robust

pessimistic?

carefully chosen
data model

get strong results

too stylized/optimistic?

define data model, then give algorithms for data from that model

danger: may overfit to the model

get best of both worlds?

semi-random models

[Molinaro Kesselhiem 19]
[Bradac G. Singla Zuzic 19]
[Garg Kale Rohwedder Svensson 19]

Input first drawn from some (stochastic) data model

Then adversary corrupts in some (bounded) way

E.g., max-finding (secretary setting)

G "green" items appear according to the model

but adversary can inject R red items in worst-case ways

get (at least) pre-corruption value?

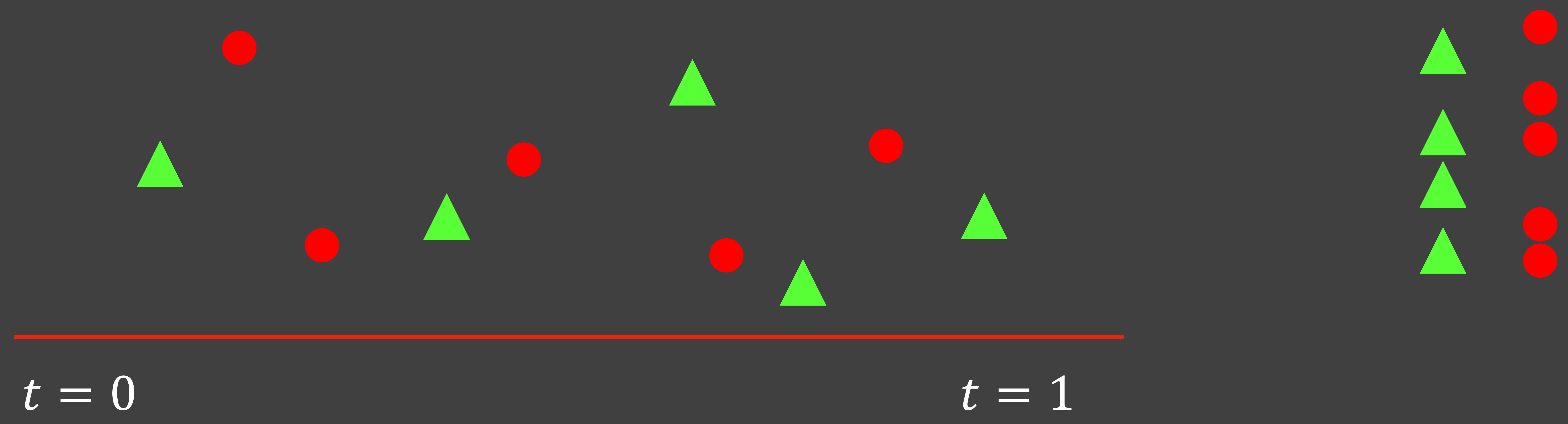
[Bradac G. Singla Zuzic 19]

byzantine max-K-finding

Adversary chooses values for G green and R red items

Adversary chooses times in $[0,1]$ for each red item
green items appear at random times in $[0,1]$

we don't see colors, want value $\approx$ sum of top K green items



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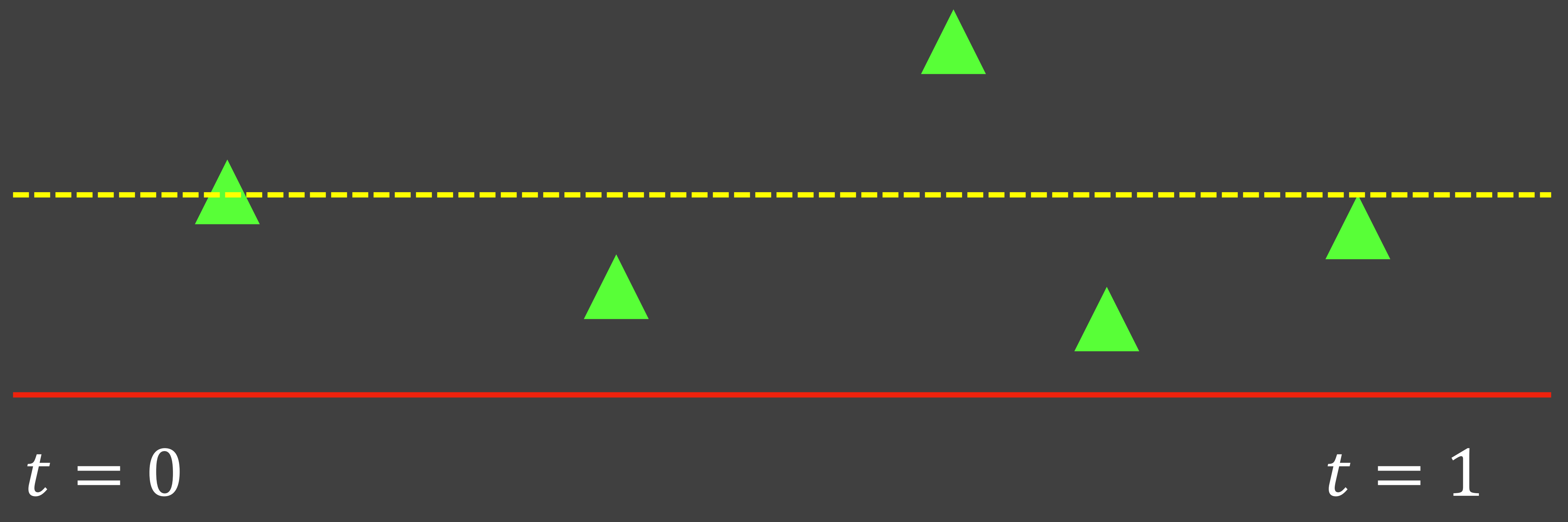

recall algorithm
without corruptions?

[Bradac G. Singla Zuzic 19]

byzantine max-K-finding

Adversary chooses values for G green and R red items

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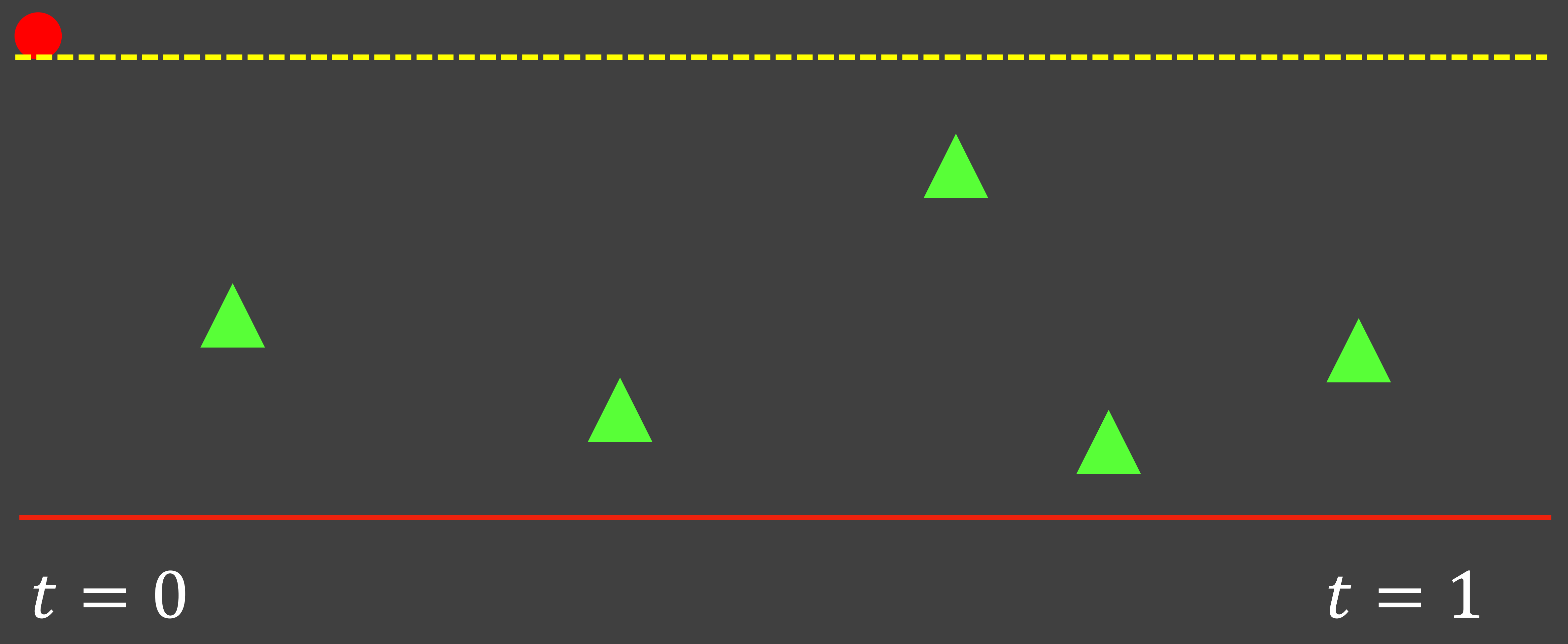


fails with corruptions!!

[Argue G. Molinaro Singla 22]

but we can still do something...

Informal Robustness Theorem:

If K is at least $\approx \log n$ and we have estimate of OPT to within poly(n)

then can achieve value $\Omega(OPT)$ even with corruptions

Good news: extends to higher-dimensional allocation problems

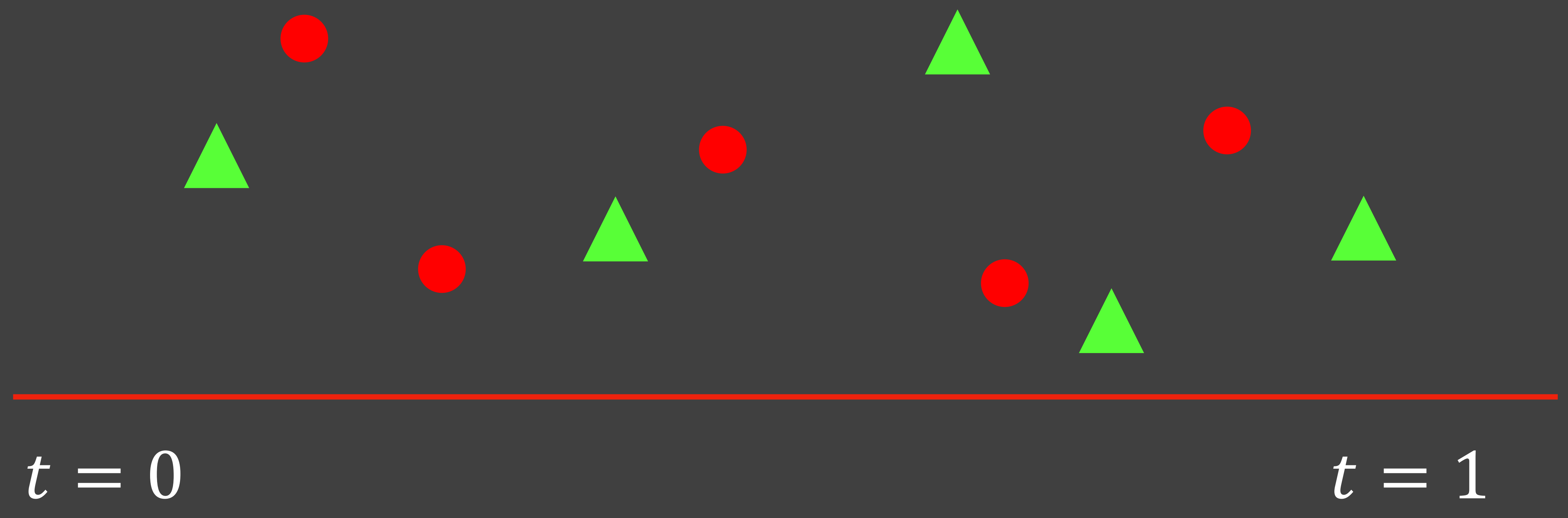
[Argue G. Molinaro Singla 22]

robust algorithmic thinking

1. Show "robust" single-parameter algorithm:
   if parameter chosen right ⇒ get good value even after corruption

2. Learn right parameter setting "robustly"

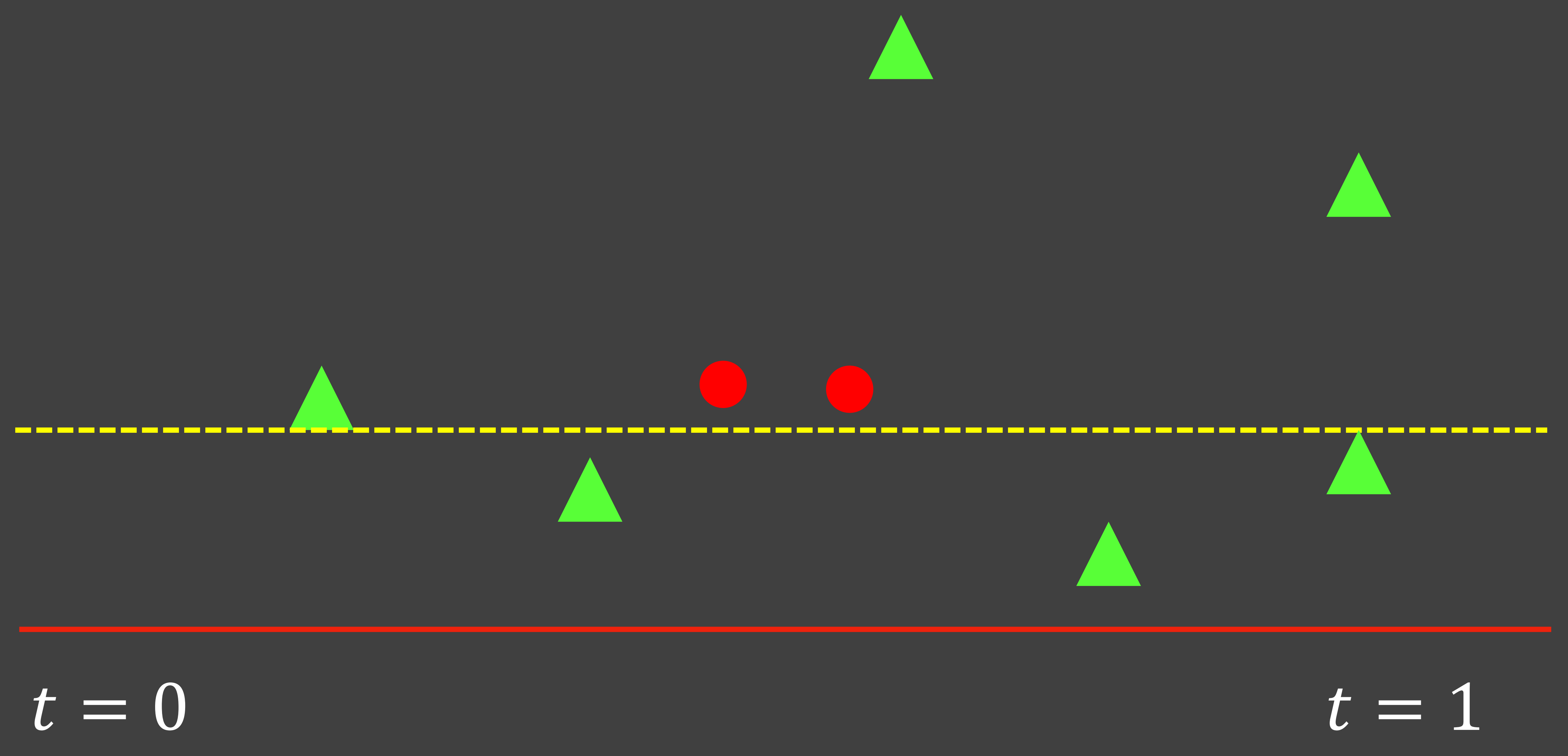
which
single-parameter
algorithm?



what threshold? idea #1

Suppose green values are $g_1 > g_2 > \ldots > g_n$

Idea #1: pick items at least threshold $T^* = g_k$



is this robust to injecting bad items??

Imagine
$g_1 = \ldots = g_{k-1} = M$
$g_k = 1$
inject reds of value 1

idea #2: a robust threshold

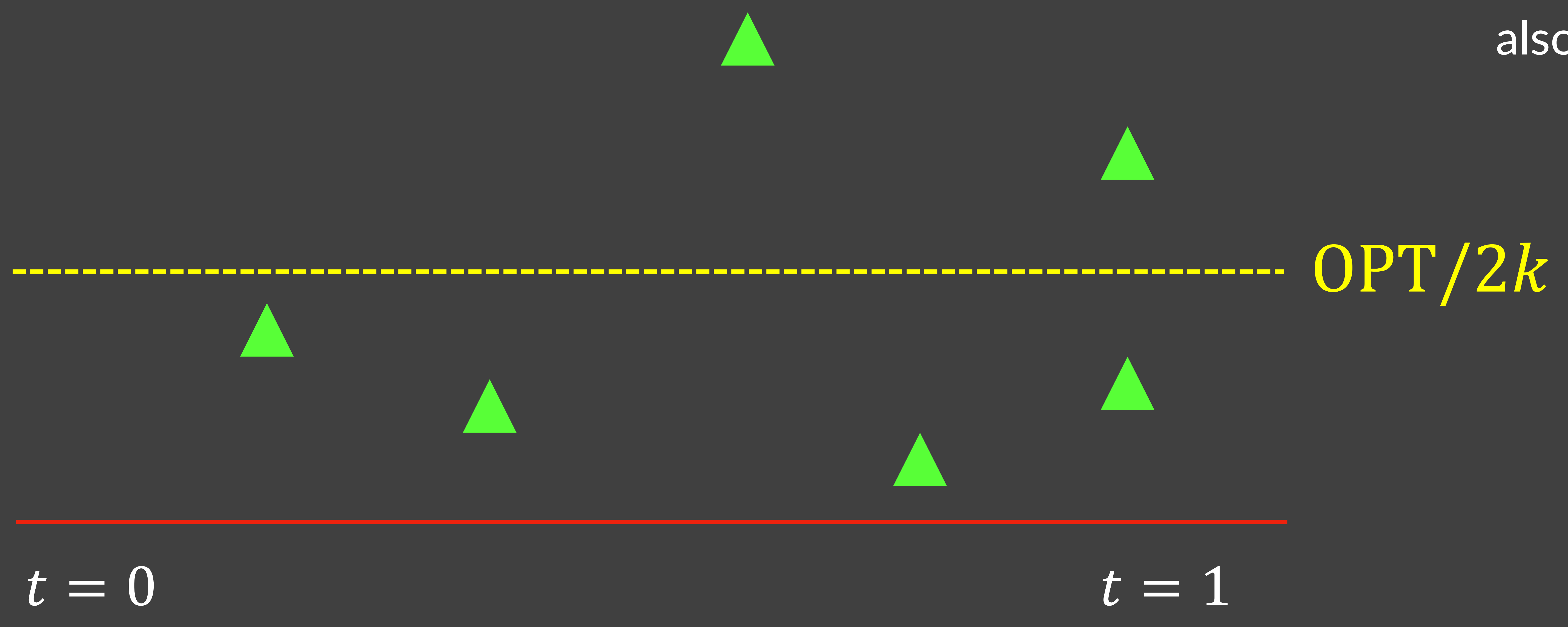
Suppose green values are $g_1 > g_2 > \ldots > g_n$

Idea #2: pick items at least threshold $T^* = \text{OPT}/2k$

Adding red items does not hurt…

$\exists$ good solution:
OPT "loses" at most
$\frac{OPT}{2k} \cdot k \leq \frac{OPT}{2}$

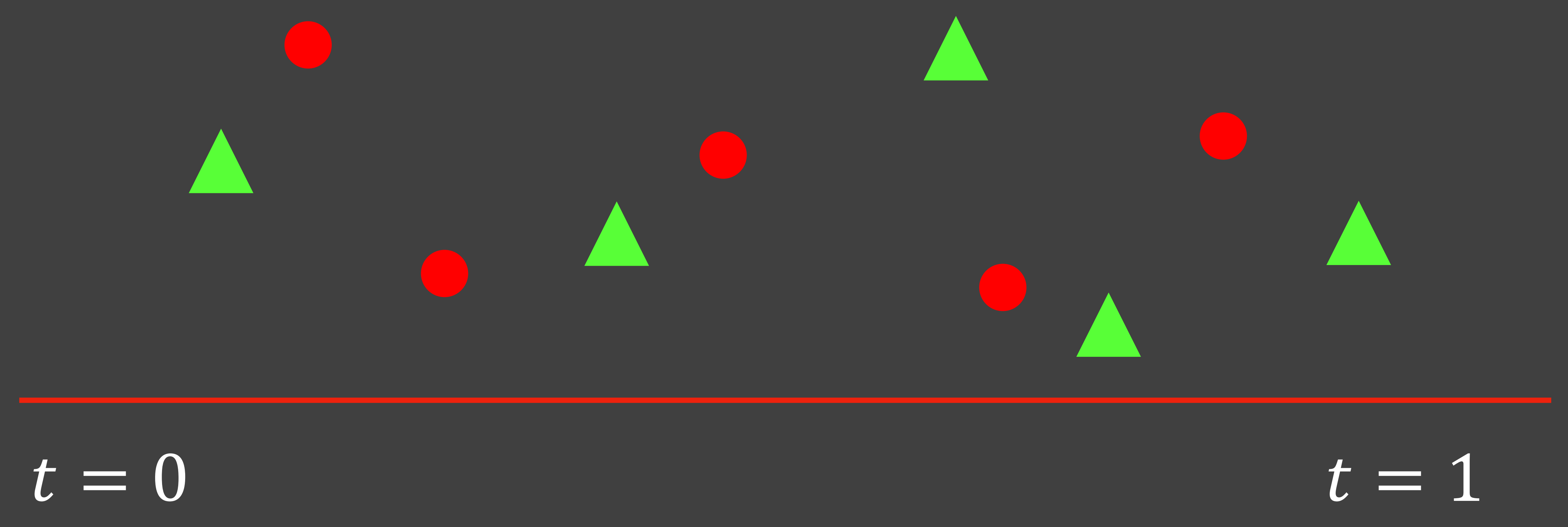
each picked red item
also gives $OPT/2k$…



robust algorithmic thinking

1. Show robust "single-parameter" algorithm:
   right threshold ⇒ get good value even after corruption

2. Learn this parameter robustly



step 2: learn threshold robustly

a. Estimate of OPT to within $\mathrm{poly}(n)$
   $\Rightarrow O(\log n)$ different guesses for OPT, need to choose right guess

b. Use online learning ("experts" algorithm) to do almost as well as best one

---

Break time [0,1] into T intervals

Use feedback from each interval to choose guess for next interval

$$\text{Payoff} \geq \Omega(OPT) - \underbrace{\sqrt{T \times \log \#\text{experts}} \times \left(\frac{OPT}{T}\right)}_{}$$

small if T is large. But want measure concentration, so T not too large!

[Argue G. Molinaro Singla 22]

our result…

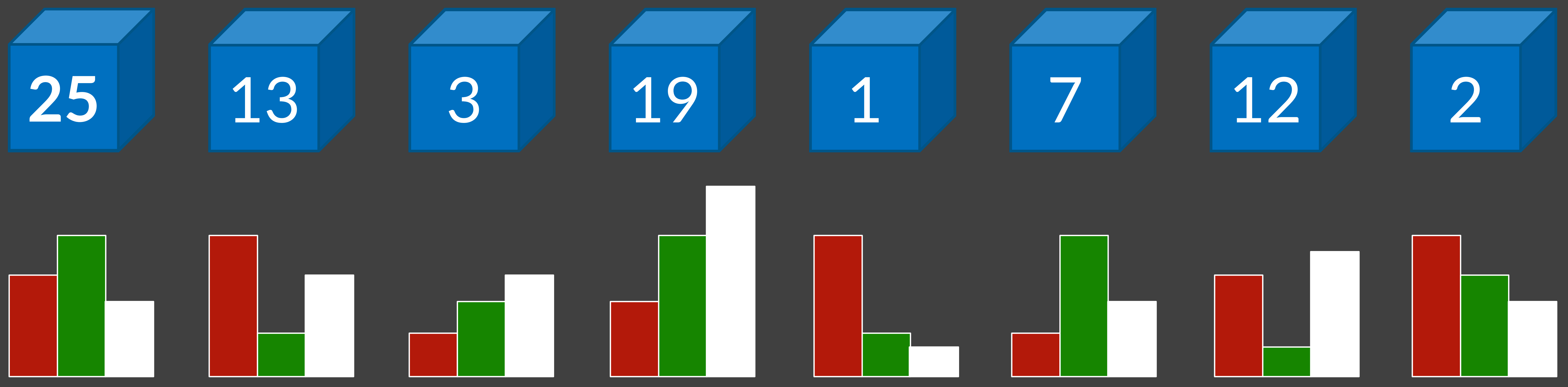
Informal Robustness Theorem:

If $K \geq O(\log n \log\log n)$ and we have estimate of OPT to within poly(n)

then we can achieve value $\Omega(OPT)$ even with corruptions

Good news: extends to higher-dimensional allocation problems

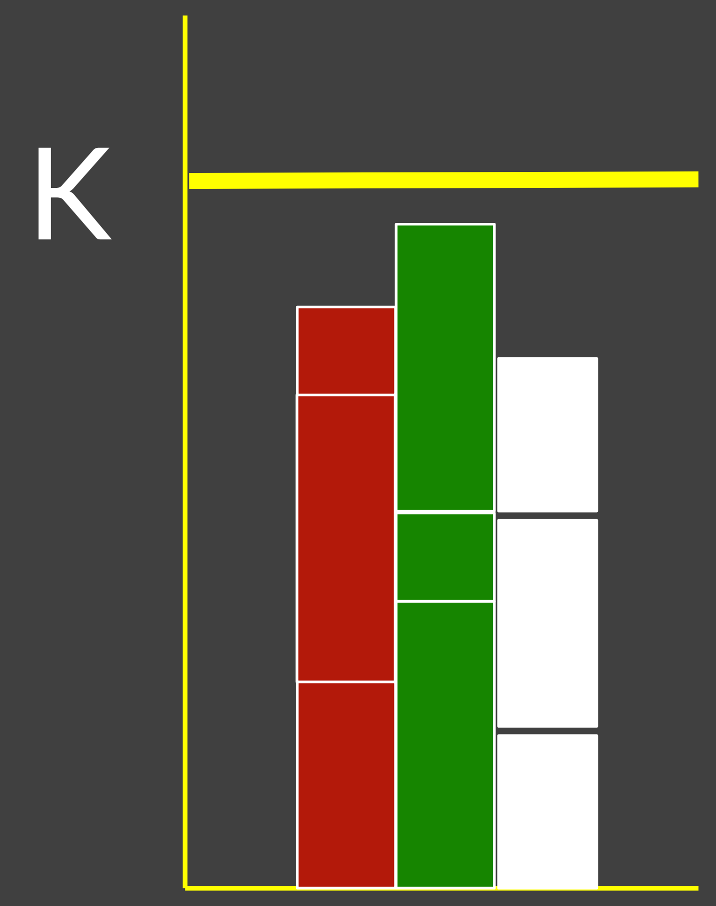
Online Allocation



Value

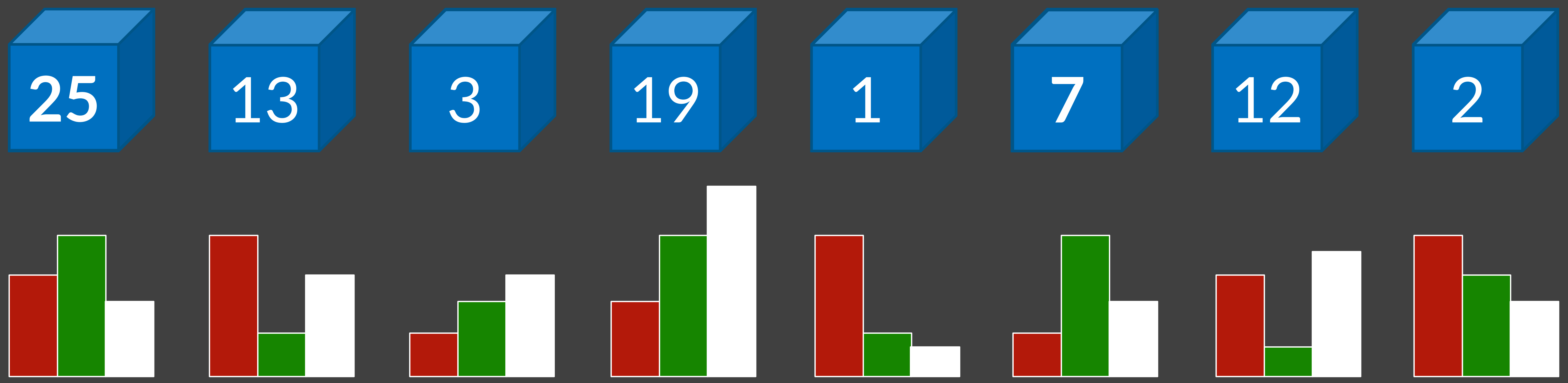
25 +13 + 7

Budget



[Agrawal Wang Ye 09]

Online Allocation



$$\max_{x \in \{0,1\}^n} v \cdot x$$

$$Ax \leq K\mathbf{1}$$

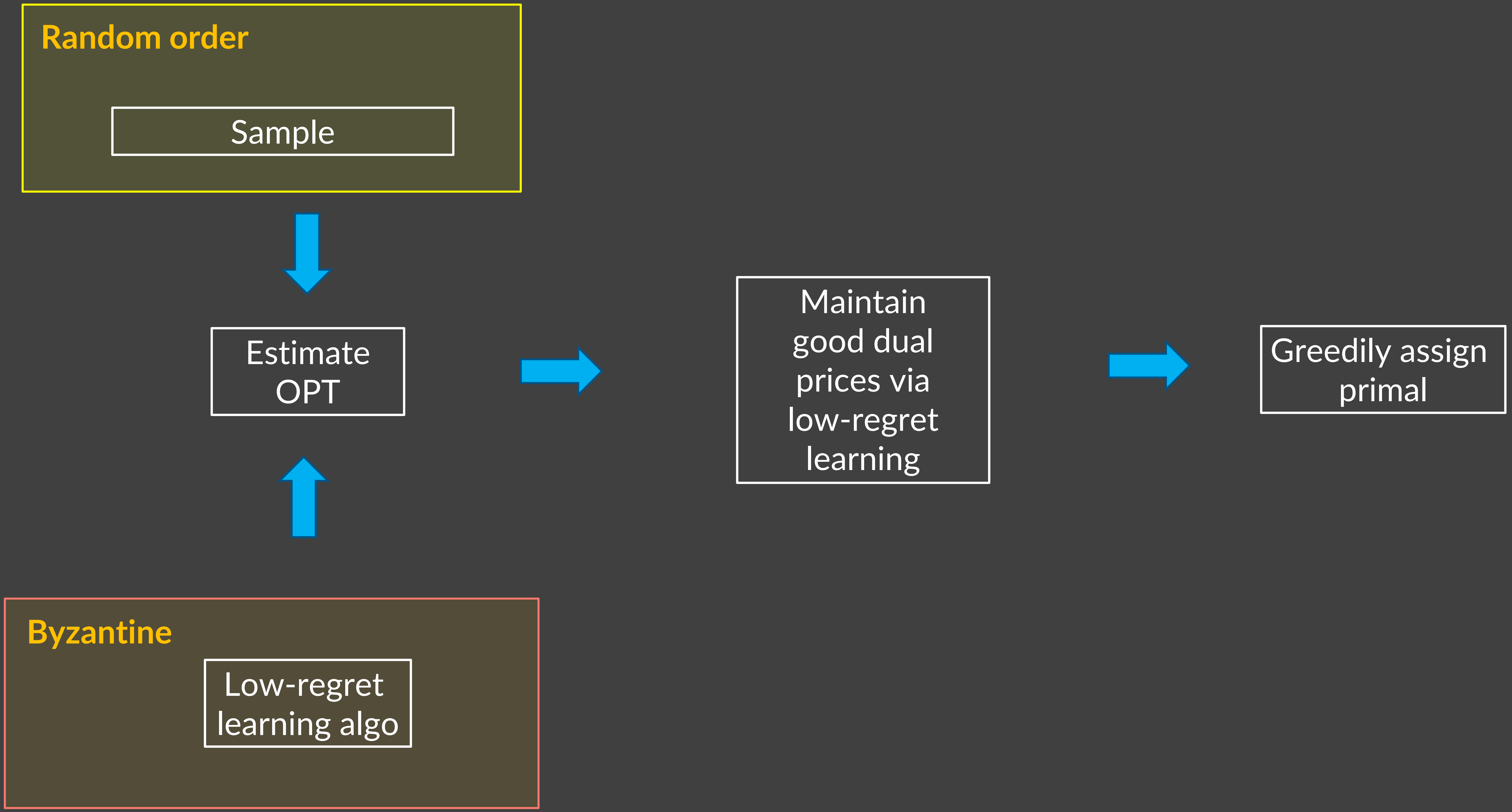
columns of A appear online

Assume: $A \in [0,1]^{d \times n}$
and $K \gg 1$

Want smallest K to get $(1+\varepsilon)$-apx for value

[Argue G. Singla Molinaro 22]

packing with corruptions



rest of today's menu…

models to go beyond worst-case:	max-finding, spanning tree, set cover
but don't overfit to these models…:	max-k-finding
and perhaps use predictions…:	paging/caching

(ML-based) predictions…

Use predictions to get better algorithms?

E.g., for caching in memory systems, suppose predict furthest-in-future page

+ If predictions perfect, then get optimal #page faults (a.k.a. Belady’s rule)

- what if predictions are correct only 10% of the time?

[G. Panigrahi Subercaseaux Sun 22]

caching with predictions

Informal Theorem:

If predict furthest-in-future page with constant probability

(and no other page predicted too often)

then get constant-competitive paging.

Q: “right” prediction model? Sample complexity of learning?

today we saw…

models to go beyond worst-case:	max-finding, spanning tree, set cover
but don’t overfit to these models…:	max-k-finding
and perhaps use predictions…:	paging/caching

to summarize

the worst-case analysis of algorithms has served us well

but we should also look beyond these robust/pessimistic guarantees

+ when do our algorithms outperform these worst-case bounds?

+ what if the input is stochastic?

+ are we over-fitting to the stochastic model?

+ can we train some model and then use its predictions?

+ ...

Thanks!